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Branching rules for typical and atypical representations of gl(n|1)

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Abstract. The $\mathrm{gl}(n|1)\downarrow \mathrm{gl}(n)\oplus \mathrm{gl}(1)$ branching rules are determined for all finite-dimensional irreducible typical and atypical representations of $\mathrm{gl}(n|1)$, using a recently introduced induced module construction for atypical modules, and confirm those found recently by Palev by another method. The lowest weights and characters of the irreducible representations are found, and the validity for $\mathrm{gl}(n|1)$ of the character formula conjectured by Hughes and King is confirmed. Some generalisations and extensions of the method are discussed.

1. Introduction

Lie superalgebras and their representations play an important role in the understanding and exploitation of supersymmetry in physical systems. This was first recognised in the area of elementary particle physics (Wess and Zumino 1974) but more recent work, particularly in nuclear physics (Iachello 1980) and condensed matter physics (Parisi and Sourlas 1979, Nambu 1985, Montorsi *et al* 1987), demonstrates the importance of supersymmetry and Lie superalgebras in a variety of other areas. A comprehensive review of the subject is given in Kostelecky and Campbell (1985).

Those Lie superalgebras whose theory and classification follow most closely that of the simple finite-dimensional Lie algebras are the basic classical superalgebras, whose properties were first developed in the definitive work of Kac (1977, 1978). While these simple Lie superalgebras have many properties in common with simple Lie algebras, there are important differences and many aspects of their representation theory remain unexplored. An important development in this respect, due to Kac (1978), was the characterisation of finite-dimensional irreducible representations into typical and atypical types. This characterisation is useful since typical representations have many properties in common with finite-dimensional representations of simple Lie algebras, and in particular are given explicitly by an induced module construction which allows a straightforward determination of their characters and dimensions (Kac 1978). The situation with atypical representations, however, is far more complex and they are still far from well understood, although progress has been made with the introduction of supertableau methods (Dondi and Jarvis 1981, Farmer and Jarvis 1983, 1984, Balantekin and Bars 1981, 1982, King 1983, Hurni 1987) and methods based on shift operators and weight space techniques (Hughes 1981, Van der Jeugt 1984, 1987, Hurni and Morel 1982, 1983).

In this paper we are concerned with the structure of finite-dimensional irreducible representations of the Lie superalgebra $\mathrm{gl}(n|1)$, $n > 1$, which for our purposes is easier to work with than the basic classical superalgebra $\mathrm{sl}(n|1)$. This problem has been considered recently by Palev (1987, 1988a, b), who has found the matrix elements of the generators in an orthogonal Gel'fand–Tsetlin basis, for each finite-dimensional irreducible representation of $\mathrm{gl}(n|1)$. We mention also earlier discussions of $\mathrm{su}(n|1)$ by Ne'eman and Sternberg (1980), Thierry-Mieg and Morel (1981), Sun and Han (1981), Thierry-Mieg (1984) and Delduc and Gourdin (1984). Our approach to the determination of the branching rules is more direct, making use of the modified induced module construction recently introduced by one of us (Gould 1989a), and avoiding the need to factor out Kac modules by their unique maximal submodules, in the case of atypical representations. We also avoid the need to introduce a particular basis and the calculation of matrix elements. On the other hand, our approach does not provide us with the extra information implicit in a complete determination of matrix elements. It would certainly be possible to extend our approach to the determination of matrix elements, following the introduction of a Gel'fand–Tsetlin basis, but we have not pursued that in view of Palev's results. It is hoped that our approach can be extended to solve the $\mathrm{gl}(m|n)\downarrow\mathrm{gl}(m)\oplus\mathrm{gl}(n)$ branching problem in particular, since the modified induced module construction is applicable to all type-I superalgebras (in Kac's (1978) notation). An approach to this problem through a direct determination of matrix elements may well be impracticable.

In what follows the $\mathrm{gl}(n|1)\downarrow\mathrm{gl}(n)\oplus\mathrm{gl}(1)$ branching rules are obtained for all irreducible typical and atypical representations. The method makes use of the $\mathrm{gl}(n)$ tensor operator shift-component formalism, first introduced by Green (1971), which is shown to play a natural role in the representation theory of atypical as well as typical modules. The branching rule obtained implicitly contains all information on the weight-space structure of typical and atypical modules: in particular we are able to determine the minimal weights of all irreducible representations. We are also able to calculate the character of an arbitrary finite-dimensional irreducible representation of $\mathrm{gl}(n|1)$, and thereby to confirm the validity for $\mathrm{gl}(n|1)$ of the character formula conjectured by Hughes and King (1987), as distinct from that obtained by Bernstein and Leites (1980) and Van der Jeugt (1987). The identification of the correct character formula has remained uncertain since Bernstein and Leites originally claimed, erroneously, that their formula is valid for all irreducible representations of $\mathrm{gl}(m|n)$ (Hughes and King 1987, Leites 1987). It remains possible that the Bernstein–Leites–Van der Jeugt formula is indeed valid for $\mathrm{gl}(n|1)$, in which case it must be equivalent for $\mathrm{gl}(n|1)$ to the Hughes–King formula, but a proof of such equivalence is not available.

Further applications and possible extensions of our methods are discussed in the final section of the paper.

2. Preliminaries

The generators of the Lie superalgebra $\mathrm{gl}(n|1)$ are given by the even $\mathrm{gl}(n)\oplus\mathrm{gl}(1)$ generators $a^i_j (1\leqslant i,\ j\leqslant n)$, Ω respectively, together with the odd generators Ψ^i, Ψ_i $(1\leqslant i\leqslant n)$, satisfying the commutation and anticommutation relations:

$$[a^i_j, a^k_l]_- = \delta^k_j a^i_l - \delta^i_l a^k_j \qquad [a^i_j, \Omega]_- = 0 \tag{1a}$$

$$[a^i_j, \Psi^k]_- = \delta^k_j \Psi^i \qquad [a^i_j, \Psi_k]_- = -\delta^i_k \Psi_j \tag{1b}$$

$$[\Omega, \Psi^k]_- = -\Psi^k \qquad [\Omega, \Psi_k]_- = \Psi_k \tag{1c}$$

$$[\Psi^k, \Psi^l]_+ = [\Psi_k, \Psi_l]_+ = 0 \qquad [\Psi^k, \Psi_l]_+ = \delta^k_l \Omega + a^k_l \tag{1d}$$

where $[\,,\,]_-$ ($[\,,\,]_+$) denotes the commutator (anticommutator): these two cases are taken into account below by the graded bracket, denoted $[\,,\,]$. We note that (1*a*) expresses the usual $\mathrm{gl}(n)\oplus\mathrm{gl}(1)$ commutation relations whilst (1*b*) expresses the fact that the operators Ψ^i (Ψ_i) transform as a vector (contragradient vector) operator of $\mathrm{gl}(n)$.

A basis for a Cartan subalgebra of $\mathrm{gl}(n|1)$ consists of the commuting operators $a^i_i (1 \leq i \leq n)$ and Ω, whose eigenvalues serve to label the weights of the representations. We denote the weights Λ of $\mathrm{gl}(n|1)$ by (notation as in Kac (1978))

$$\Lambda = \sum_{i=1}^{n} \Lambda_i \varepsilon_i + \omega\delta_1 = (\Lambda_1, \ldots, \Lambda_n|\omega) \tag{2}$$

so that, with this convention, the root system of $\mathrm{gl}(n|1)$ is given by the set of even roots $\pm(\varepsilon_i - \varepsilon_j)$ $(1 \leq i < j \leq n)$ together with the set of odd roots $\pm(\varepsilon_i - \delta_1)$ $(1 \leq i \leq n)$. Following Kac (1978) we choose, as a system of simple roots, the distinguished set

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1} \qquad 1 \leq i < n \qquad \alpha_s = \varepsilon_n - \delta_1$$

so that the sets of even and odd positive roots are given respectively by

$$\Phi_0^+ = \{\varepsilon_i - \varepsilon_j | 1 \leq i < j \leq n\} \qquad \Phi_1^+ = \{\varepsilon_i - \delta_1 | 1 \leq i \leq n\}$$

and we set

$$\rho_0 = \tfrac{1}{2} \sum_{i<j}^{n} (\varepsilon_i - \varepsilon_j) = \tfrac{1}{2} \sum_{i=1}^{n} (n+1-2i)\varepsilon_i$$

$$\rho_1 = \tfrac{1}{2} \sum_{i=1}^{n} (\varepsilon_i - \delta_1) = \tfrac{1}{2} \sum_{i=1}^{n} \varepsilon_i - \tfrac{1}{2} n\delta_1$$

$$\rho = \rho_0 - \rho_1 .$$

We note that $\mathrm{gl}(n|1)$ admits a non-degenerate even invariant bilinear supertrace form arising from the fundamental vector representation π:

$$(x, y) = \mathrm{str}(\pi(x)\pi(y)) \qquad x, y \in \mathrm{gl}(n|1)$$

leading to

$$(\varepsilon_i, \varepsilon_j) = \delta_{ij} \qquad (\varepsilon_i, \delta_1) = 0 \qquad (\delta_1, \delta_1) = -1.$$

This in turn induces a non-degenerate bilinear form on the weights, given by

$$(\Lambda, \Lambda') = \sum_{i=1}^{n} \Lambda_i \Lambda'_i - \omega\omega' \tag{3}$$

with Λ as in (2) and $\Lambda' = (\Lambda'_1, \ldots, \Lambda'_n|\omega')$.

Every finite-dimensional irreducible $\mathrm{gl}(n|1)$ module admits a unique (up to scalar multiples) weight vector v^Λ of weight Λ satisfying the conditions

$$a^i_j v^\Lambda = 0 \qquad 1 \leq i < j \leq n \qquad \Psi^i v^\Lambda = 0 \qquad 1 \leq i \leq n. \tag{4}$$

Such a vector is called a highest-weight vector and its weight Λ, called the highest weight of the representation, uniquely characterises the representation (Kac 1978): throughout we denote the irreducible $\mathrm{gl}(n|1)$ module with highest weight Λ by $V(\Lambda)$. The components of the highest weight Λ necessarily satisfy the conditions

$$\Lambda_i - \Lambda_j \in \mathbb{Z}^+ \qquad 1 \leq i < j \leq n \tag{5}$$

with ω an arbitrary constant, which are just the conditions that Λ constitute a dominant integral weight of the Lie algebra $\mathrm{gl}(n)\oplus\mathrm{gl}(1)$: throughout we denote the finite-dimensional irreducible module over $\mathrm{gl}(n)\oplus\mathrm{gl}(1)$ with highest weight Λ by $V_0(\Lambda)$. We denote the set of all weights Λ, whose components satisfy (5), by D^+ (the set of dominant integral weights).

Corresponding to every $\Lambda\in D^+$ we may construct an indecomposable finite-dimensional $\mathrm{gl}(n|1)$ module with highest weight Λ using the induced module construction of Kac (1978). To this end we find it convenient for the moment to denote the Lie superalgebra $\mathrm{gl}(n|1)$ simply by L and to let L_0 denote the Lie subalgebra $\mathrm{gl}(n)\oplus\mathrm{gl}(1)$. We let L_+ (L_-) denote the graded-Abelian subalgebra spanned by the operators Ψ^i (Ψ_i) giving the consistent $\mathbb{Z}$-gradation

$$L=L_-\oplus L_0\oplus L_+. \tag{6}$$

We denote the universal enveloping algebras of L, L_0, $L_\pm$ by U, U_0, $U_\pm$ and we denote by $\bar{U}_\pm$ the universal enveloping algebra of the subalgebra

$$\bar{L}_\pm=L_0\oplus L_\pm. \tag{7}$$

We note that the algebras $U_\pm$ are 2^n-dimensional with basis consisting of $1\in\mathbb{C}$ together with all basis monomials

$$\Psi^{i_1}\Psi^{i_2}\ldots\Psi^{i_k}(\text{resp } \Psi_{i_1}\Psi_{i_2}\ldots\Psi_{i_k}) \qquad 1\leqslant i_1<\ldots<i_k\leqslant n,\, 1\leqslant k\leqslant n.$$

From the Poincare–Birkhoff–Witt theorem (Scheunert 1979) we may write, in view of (6) and (7):

$$\begin{aligned} U&=U_-U_0U_+=U_+U_0U_-\\ &=U_-\bar{U}_+=U_+\bar{U}_-. \end{aligned} \tag{8}$$

Given a finite-dimensional irreducible L_0-module $V_0(\Lambda)$, we turn $V_0(\Lambda)$ into a $\bar{U}_+$ module by defining

$$L_+V_0(\Lambda)=(0). \tag{9}$$

The induced L-module $\bar{V}(\Lambda)$ is then defined by (Kac 1978)

$$\bar{V}(\Lambda)=U_-\otimes_{\bar{U}_+}V_0(\Lambda)=\bigoplus_{1\leqslant i_1<\ldots<i_k\leqslant n}\Psi_{i_1}\ldots\Psi_{i_k}\otimes V_0(\Lambda) \tag{10}$$

which constitutes an indecomposable L-module with highest weight Λ and dimension $\dim\bar{V}(\Lambda)=2^n\dim V_0(\Lambda)$. In a similar way we may define

$$L_-V_0(\Lambda)=(0) \tag{11}$$

which leads to the induced L-module

$$\bar{V}_-(\Lambda)=U_+\otimes_{\bar{U}_-}V_0(\Lambda). \tag{12}$$

This is also indecomposable, but in this case is cyclically generated by a lowest-weight vector of weight Λ_-, where Λ_- is the lowest weight of $V_0(\Lambda)$: recall that $\Lambda_-=(\Lambda_n,\Lambda_{n-1},\ldots,\Lambda_1|\omega)$. We have the following result.

Theorem 1 (Kac 1978). $\bar{V}(\Lambda)=V(\Lambda)$ (i.e. $\bar{V}(\Lambda)$ is irreducible) if and only if $(\Lambda+\rho,\alpha)\neq 0$, for all $\alpha\in\Phi_1^+$.

A weight $\Lambda \in D^+$ satisfying the conditions of theorem 1, and the corresponding module $V(\Lambda)$, are said to be *typical*. Similarly, Λ and $V(\Lambda)$ are called *atypical* if $(\Lambda+\rho, \alpha)=0$ for some $\alpha \in \Phi_1^+$. We note that this atypicality condition is equivalent to the condition that $\Lambda_i+\omega=i-n$ for a corresponding value of i, $1 \leqslant i \leqslant n$; it follows from (5) that for $\Lambda \in D^+$ this condition holds for, at most, one such value. (We say that the irreducible representations of gl$(n|1)$ are at most singly atypical.) The structure of typical irreducible L-modules follows immediately from the induced module construction (10). In particular the gl$(n|1)\downarrow$gl$(n)\oplus$gl(1) branching rules for typical representations may be determined by exploiting the fact that the operators

$$\Psi_{i_1}\Psi_{i_2}\ldots\Psi_{i_k} \qquad 1 \leqslant i_1 < i_2 \ldots < i_k \leqslant n \tag{13}$$

form basis vectors of the antisymmetric contragredient kth rank tensor representation of gl$(n)\oplus$gl(1), which follows directly from the anticommutation relations (1_d): the operators (13) form a basis for the irreducible gl$(n)\oplus$gl(1) module with highest weight $(\dot{0}, -\dot{1}_k|k)$. We have, from well known tensor product rules (see, for example, Gould 1989b)

$$V_0(\dot{0}, -\dot{1}_k|k) \otimes V_0(\Lambda) = \bigoplus_{1 \leqslant i_1 < \ldots < i_k \leqslant n}' V_0(\Lambda - (\varepsilon_{i_1} - \delta_1) - \ldots - (\varepsilon_{i_k} - \delta_1))$$

where the decomposition on the RHS is multiplicity free and the sum is over all ascending sequences of k integers $1 \leqslant i_1 < \ldots < i_k \leqslant n$ (corresponding to the weights in $V_0(\dot{0}, -\dot{1}_k|k)$), subject to the condition that

$$\Lambda - \sum_{r=1}^{k} (\varepsilon_{i_r} - \delta_1) \in D^+.$$

By this means we arrive at the following branching rule for the decomposition of a Kac module $\bar{V}(\Lambda)$ into irreducible gl$(n)\oplus$gl(1) modules:

$$\bar{V}(\Lambda) = \bigoplus_{\Lambda_0} V_0(\Lambda_0)$$

where the sum is over all gl$(n)\oplus$gl(1) highest weights $\Lambda_0 = (\Lambda_{0_1}, \ldots, \Lambda_{0_n}|\omega_0)$ whose components satisfy the conditions

$$\begin{aligned} &\omega_0 = \omega + \sum_{i=1}^{n} (\Lambda_i - \Lambda_{0_i}) \\ &\Lambda_i \geqslant \Lambda_{0_i} \geqslant \Lambda_i - 1 \qquad 1 \leqslant i \leqslant n \\ &\Lambda_{0_i} \geqslant \Lambda_{0_{i+1}} \qquad 1 \leqslant i \leqslant n-1 \end{aligned} \tag{14}$$

and have $(\Lambda_i - \Lambda_{0_i})$ integral, $1 \leqslant i \leqslant n$.

In the case that $V(\Lambda)$ is a typical gl$(n|1)$ module, and so equal to $\bar{V}(\Lambda)$, the gl$(n|1)\downarrow$gl$(n)\oplus$gl(1) branching rules follow immediately from (14). In the case that $V(\Lambda)$ is atypical, however, the situation is more complex, since in such a case the Kac module $\bar{V}(\Lambda)$ is no longer irreducible and $V(\Lambda)$ only appears as a subquotient of $\bar{V}(\Lambda)$:

$$V(\Lambda) \simeq \bar{V}(\Lambda)/M(\Lambda)$$

where $M(\Lambda)$ is the (unique) maximal submodule of $\bar{V}(\Lambda)$. However, for arbitrary $\Lambda \in D^+$, we may employ the result (Gould 1989a) that the lowest-weight Kac module (cf (11) and (12))

$$\bar{V}_-(\Lambda - 2\rho_1) = U_+ \otimes_{\bar{U}_-} V_0(\Lambda - 2\rho_1) \qquad L_- V_0(\Lambda - 2\rho_1) = (0) \tag{15}$$

contains, as a unique irreducible L-submodule,

$$V(\Lambda)=U[\bar{\Psi}\otimes V_0(\Lambda-2\rho_1)]=U_-[\bar{\Psi}\otimes V_0(\Lambda-2\rho_1)] \tag{16}$$

where

$$\bar{\Psi}=\Psi^1\Psi^2\ldots\Psi^n. \tag{17}$$

We note, since $(\rho_1,\alpha)=0$ for $\alpha\in\Phi_0^+$, that $\Lambda\in D^+$ if and only if $\Lambda-2\rho_1\in D^+$.

It is our aim in the following to employ the modified induced module construction of (16) in order to determine the $\mathrm{gl}(n|1)\downarrow\mathrm{gl}(n)\oplus\mathrm{gl}(1)$ branching rules for *all* irreducible $\mathrm{gl}(n|1)$ modules $V(\Lambda)$, $\Lambda\in D^+$. We expect, in the case of atypical $\Lambda\in D^+$, a modification to the brandhing rule (14), with the deletion of certain $\mathrm{gl}(n)\oplus\mathrm{gl}(1)$ highest weights that satisfy the 'lexicality' conditions given there.

3. Antisymmetric tensors

We introduce the alternating symbol $\varepsilon_{pq\ldots v}$, which is completely antisymmetric in its n subscripts, and which has the value 1 when $p=1$, $q=2,\ldots,v=n$. Then we can write the operator $\bar{\Psi}$ of (17) in the form

$$\bar{\Psi}=(1/n!)\varepsilon_{pq\ldots v}\Psi^p\Psi^q\ldots\Psi^v \tag{17'}$$

(summation convention over repeated indices here and below).

It is convenient to generalise by introducing also $\bar{\Psi}_{pqr\ldots s}$, antisymmetric in its k subscripts, and defined for each $k=0,1,2,\ldots,n$ as (cf Gould 1988a)

$$\bar{\Psi}_{pqr\ldots s}=[(-1)^{k(k-1)/2}/(n-k)!]\varepsilon_{pqr\ldots stu\ldots v}\Psi^t\Psi^u\ldots\Psi^v. \tag{18}$$

These operators satisfy

$$\begin{aligned}\Psi^m\bar{\Psi}_{pqr\ldots s}&=(-1)^{n-k}\bar{\Psi}_{pqr\ldots s}\Psi^m\\ &=\delta_p^m\bar{\Psi}_{qr\ldots s}-\delta_q^m\bar{\Psi}_{pr\ldots s}+\cdots+(-1)^{k-1}\delta_s^m\bar{\Psi}_{pqr\ldots}\end{aligned} \tag{19a}$$

and in particular

$$\begin{gathered}\Psi^m\bar{\Psi}=0 \qquad \Psi^m\bar{\Psi}_p=\delta_p^m\bar{\Psi}\\ \Psi^m\bar{\Psi}_{pq}=\delta_p^m\bar{\Psi}_q-\delta_q^m\bar{\Psi}_p.\end{gathered} \tag{19b}$$

We note that $\bar{\Psi}_{pqr\ldots s}$ is a kth rank, contragredient, antisymmetric, pseudotensor operator with respect to $\mathrm{gl}(n)\oplus\mathrm{gl}(1)$: the commutation relations with the generators are found by repeated application of (1) to be

$$[a_j^i,\bar{\Psi}_{pqr\ldots s}]=\delta_j^i\bar{\Psi}_{pqr\ldots s}-\delta_p^i\bar{\Psi}_{jqr\ldots s}-\cdots-\delta_s^i\bar{\Psi}_{pqr\ldots j} \tag{20a}$$

$$[\Omega,\bar{\Psi}_{pqr\ldots s}]=(k-n)\bar{\Psi}_{pqr\ldots s}. \tag{20b}$$

To appreciate the significance of the tensors (18) we note, in view of the modified induced module construction (16), that the irreducible L-module $V(\Lambda)$ is spanned by the vectors

$$\Psi_{i_1}\Psi_{i_2}\ldots\Psi_{i_k}(\bar{\Psi}\otimes v) \qquad 1\leqslant i_1<i_2<\ldots<i_k\leqslant n \qquad v\in V_0(\Lambda-2\rho_1)$$

where, by definition, we have

$$\Psi_iV_0(\Lambda-2\rho_1)=(0) \qquad 1\leqslant i\leqslant n. \tag{21}$$

In view of the above, we deduce immediately that for $v \in V_0(\Lambda - 2\rho_1)$

$$\Psi_{i_1}\Psi_{i_2}\ldots\Psi_{i_k}(\bar{\Psi}\otimes v) = \Psi_{i_1}\ldots\Psi_{i_{k-l}}[\Psi_{i_k}, \bar{\Psi}]\otimes v$$

where we necessarily have

$$[\Psi_i, \bar{\Psi}] = \bar{\Psi}_j \otimes C_i^j \tag{22}$$

for suitable coefficients C_i^j, which must belong to the enveloping algebra of gl$(n)\oplus$gl(1). To determine these coefficients we multiply on the left by Ψ^l from which we obtain, in view of (19),

$$\begin{aligned}\bar{\Psi}\otimes C_i^l &= \Psi^l[\Psi_i, \bar{\Psi}]\\ &= -\{[\Psi_i, \Psi^l\bar{\Psi}] - [\Psi_i, \Psi^l]\bar{\Psi}\}\\ &= (\delta_i^l\Omega + a_l^i)\bar{\Psi} = \bar{\Psi}\otimes(a+\Omega-n+1)_l^i\end{aligned}$$

where in the second equation we used the result $\Psi^i\bar{\Psi} = \bar{\Psi}\Psi^i = 0$ as in (19). By this means we deduce that

$$C_i^l = (a+\Omega-n+1)_i^l = a_i^l + \delta_i^l(\Omega-n+1)$$

and substituting into (22) we arrive at

$$[\Psi_i, \bar{\Psi}] = \bar{\Psi}_j\otimes(a+\Omega-n+1)_i^j. \tag{23}$$

In a similar way we deduce the result

$$[\Psi_i, \bar{\Psi}_j] = \bar{\Psi}_{lk}\otimes C_{ij}^{lk} \tag{24}$$

for suitable coefficients C_{ij}^{lk} belonging to the enveloping algebra of gl$(n)\oplus$gl(1). In this case we multiply (24) on the left by Ψ^q to give, using (19*a*)

$$\begin{aligned}2\bar{\Psi}_k\otimes C_{ij}^{qk} &= \Psi^q[\Psi_i, \bar{\Psi}_j]\\ &= -\{[\Psi_i, \Psi^q\bar{\Psi}_j] - [\Psi_i, \Psi^q]\bar{\Psi}_j\}\\ &= (a+\Omega)_i^q\bar{\Psi}_j - \delta_j^q[\Psi_i, \bar{\Psi}]\end{aligned}$$

where we have employed the antisymmetry of the coefficients C. Using (21) and the commutation relations (20) we then obtain

$$2\bar{\Psi}_k\otimes C_{ij}^{qk} = -\delta_j^q\bar{\Psi}_k\otimes(a+\Omega-n+2)_i^k + \bar{\Psi}_j\otimes(a+\Omega-n+2)_i^q.$$

Comparing coefficients of $\bar{\Psi}_k$ we thereby obtain

$$2C_{ij}^{qk} = \delta_j^k(a+\Omega-n+2)_i^q - \delta_j^q(a+\Omega-n+2)_i^k.$$

We thus finally arrive at

$$\begin{aligned}[\Psi_i, \bar{\Psi}_j] &= \bar{\Psi}_{qk}\otimes C_{ij}^{qk}\\ &= \bar{\Psi}_{qj}\otimes(a+\Omega-n+2)_i^q.\end{aligned} \tag{25}$$

By a simple induction argument we obtain, using (19*a*) and the commutation relations of (20), the general result:

$$[\Psi_i, \bar{\Psi}_{i_1 i_2\ldots i_k}] = \bar{\Psi}_{li_1\ldots i_k}\otimes(a+\Omega+k+1-n)_i^l. \tag{26}$$

We note, from the commutation relations (20), that the result (26) may be expressed in terms of the gl(n) adjoint matrix (Green 1971, Gould 1980)

$$\bar{a}_i^{\,j} = -a_i^{\,j} \tag{27}$$

according to

$$[\Psi_i, \bar{\Psi}_{i_1 \ldots i_k}] = (\Omega + n - 1 - k - \bar{a})_i{}^l \bar{\Psi}_{l i_1 \ldots i_k}. \tag{28}$$

It follows, by repeated application of the above results, that the irreducible L-module (16) is spanned by vectors (summation over repeated indices assumed)

$$\begin{aligned}\Psi_{i_k}\Psi_{i_{k-l}} \ldots \Psi_{i_1}(\bar{\Psi}\otimes v) &= \Psi_{i_k} \ldots \Psi_{i_2}\bar{\Psi}_{j_1} \otimes (a+\Omega-n+1)_{i_1}^{j_1} v \\ &= \bar{\Psi}_{i_k \ldots j_1} \otimes (a+\Omega+k-n)_{i_k}^{j_k} \ldots \\ &\qquad (a+\Omega-n+2)_{i_2}^{j_2}(a+\Omega-n+1)_{i_1}^{j_1} v \end{aligned} \tag{29a}$$

for $v \in V_0(\Lambda - 2\rho_1)$. Alternatively, in terms of the gl(n) adjoint matrix (25) we may write

$$\Psi_{i_k}\Psi_{i_{k-1}} \ldots \Psi_{i_1}(\bar{\Psi}\otimes v) = (\Omega+n-k-\bar{a})_{i_k}{}^{j_k} \ldots (\Omega+n-1-\bar{a})_{i_1}{}^{j_1}\bar{\Psi}_{j_k \ldots j_1} \otimes v. \tag{29b}$$

The results (29) yield a great deal of information on the structure of the spaces $V(\Lambda)$. As noted previously, the vectors $\bar{\Psi}_{i_1 \ldots i_k}$ constitute the basis vectors of the antisymmetric contragredient tensor representation $(\dot{0}, -\dot{1}_k|k)$ of gl(n)$\oplus$gl(1). In this case not all irreducible representations in the tensor product

$$V_0(\dot{0}, -\dot{1}_k|k) \otimes V_0(\Lambda - 2\rho_1)$$

will occur in $V(\Lambda)$, due to the special nature of (29) which in fact imply certain vanishings (related to atypicality). This point is best illustrated in terms of tensor operator shift components, which will be discussed in § 4.

We conclude this section by noting that (29) may be expressed as

$$[\Psi_{i_k}, [\Psi_{i_{k-1}}, \ldots, [\Psi_{i_1}, \bar{\Psi}] \ldots]] = \bar{\Psi}_{j_k \ldots j_1} \otimes (a+\Omega+k-n)_{i_k}^{j_k} \ldots (a+\Omega+1-n)_{i_1}^{j_1}. \tag{30}$$

In particular, it follows from (18) that we may write

$$1 = \bar{\Psi}_{n n-1 \ldots 1} = (-1)^{\frac{1}{2}n(n-1)}\bar{\Psi}_{12 \ldots n} = \mathrm{sn}(\pi)\bar{\Psi}_{\pi(n) \ldots \pi(1)}$$

where π is any permutation of the numbers $1, \ldots, n$. Equation (30) then implies that

$$\begin{aligned}\Delta' &= [\Psi_n, [\Psi_{n-1}, \ldots, [\Psi_1, \bar{\Psi}] \ldots]] \\ &= \sum_\pi \mathrm{sn}(\pi)(a+\Omega)_n^{\pi(n)}(a+\Omega-1)_{n-1}^{\pi(n-1)} \ldots (a+\Omega+1-n)_1^{\pi(1)}\end{aligned} \tag{31}$$

where the sum is over all permutations of the numbers $1, \ldots, n$. If we set

$$\Psi = \Psi_n \Psi_{n-1} \ldots \Psi_1$$

then we may write (cf Gould (1989a), where the operator Ψ was denoted T_-)

$$\Psi\bar{\Psi} = \Delta' + \varphi \qquad \varphi \in UL_-$$

with Δ' as in (31). Following Gould (1989a), we see that Δ' necessarily belongs to the centre of the enveloping algebra of gl(n)$\oplus$gl(1), and its eigenvalue on the irreducible gl(n)$\oplus$gl(1) module $V_0(\Lambda - 2\rho_1)$ is given by

$$\chi_{\Lambda - 2\rho_1}(\Delta') = \prod_{\alpha \in \Phi_1^+} (\Lambda + \rho, \alpha)$$

which follows directly from the results of § 4.

4. Shift tensors and branching rules

Following Bracken and Green (1971) and Green (1971), the $gl(n)\oplus gl(1)$ contragredient vector operator Ψ_i may be resolved into $gl(n)\oplus gl(1)$ shift components

$$\Psi_i = \sum_{r=1}^{n} \Psi[r]_i$$

where $\Psi[r]_i$ is a contragredient vector operator that decreases the highest weight of a representation of gl(n), in a given irreducible representation of gl(n|1), by the weight ε_r, whilst increasing the gl(1) weight ω by 1 unit:

$$\Psi[r]_i v \in V_0(\Lambda_0 - \varepsilon_r + \delta_1) \qquad v \in V_0(\Lambda_0).$$

The above shift components may be constructed using

$$\Psi[r]_i = \bar{P}[r]_i{}^j \Psi_j = \Psi_j P[r]^j_i$$

where $\bar{P}[r]$, $P[r]$ denote the polynomials in the matrices $\bar{a}$, a respectively, defined by

$$\bar{P}[r] = \prod_{l\neq r}^{n} \left(\frac{\bar{a} - \bar{\alpha}_l}{\bar{\alpha}_r - \bar{\alpha}_l}\right) \qquad P[r] = \prod_{l\neq r}^{n} \left(\frac{a - \alpha_l}{\alpha_r - \alpha_l}\right).$$

Here the roots α_r and adjoint roots $\bar{\alpha}_r$ are gl(n)invariants which take constant values on an irreducible gl(n) module with highest weight $(\Lambda_{0_1}, \ldots, \Lambda_{0_n})$ given by

$$\alpha_r = \Lambda_{0_r} + n - r \qquad \bar{\alpha}_r = r - 1 - \Lambda_{0_r}. \tag{32}$$

We note that the roots α_r ($\bar{\alpha}_r$) are all distinct, on a given (finite-dimensional) irreducible gl(n) module, and the following identities hold (Green 1971):

$$\begin{aligned} \bar{a}_i{}^j \bar{P}[r]_j{}^k &= \bar{P}[r]_i{}^j \bar{a}_j{}^k = \bar{\alpha}_r \bar{P}[r]_i{}^k \\ a^i_j P[r]^j_k &= P[r]^i_j a^j_k = \alpha_r P[r]^i_k. \end{aligned} \tag{33}$$

More generally a product of two odd operators:

$$\Psi_i \Psi_j = -\Psi_j \Psi_i$$

is to determine an antisymmetric contragredient tensor operator of rank 2, with the following resolution into shift components:

$$\Psi_i \Psi_j = \sum_{r<l}^{n} \Psi[r, l]_{ij}$$

where

$$\Psi[r, l]_{ij} = \Psi[r]_i \Psi[l]_j + \Psi[l]_i \Psi[r]_j$$

which effect the following shifts on the $gl(n)\oplus gl(1)$ representation labels:

$$\Psi[r, l]_{ij} v \in V_0[\Lambda_0 - (\varepsilon_r - \delta_1) - (\varepsilon_l - \delta_1)] \qquad v \in V_0(\Lambda_0).$$

Still more generally, the antisymmetric kth-rank contragredient tensor $\Psi_{i_1}\Psi_{i_2}\ldots\Psi_{i_k}$ may be resolved into shift components

$$\Psi_{i_1} \ldots \Psi_{i_k} = \sum_{1\leq r_1 < r_2 \ldots < r_k \leq n} \Psi[r_1, r_2, \ldots, r_k]_{i_1 i_2 \ldots i_k}$$

where

$$\begin{aligned}\Psi[r_1, r_2, \ldots, r_k]_{i_1 i_2 \ldots i_k} &= \sum_\pi \Psi[r_{\pi(1)}]_{i_1} \Psi[r_{\pi(2)}]_{i_2} \ldots \Psi[r_{\pi(k)}]_{i_k} \\ &= \Psi[r_1]_{i_1} \Psi[r_2, \ldots, r_k]_{i_2 \ldots i_k} + \Psi[r_2]_{i_1} \Psi[r_1, r_3, \ldots, r_k]_{i_2 \ldots i_k} \\ &\quad + \cdots + \Psi[r_k]_{i_1} \Psi[r_1, r_2, \ldots, r_{k-l}]_{i_2 \ldots i_k} \end{aligned} \tag{34}$$

which effect the following shifts on the $\mathrm{gl}(n) \oplus \mathrm{gl}(1)$ representation labels:

$$\Psi[r_1, \ldots, r_k]_{i_1 \ldots i_k} v \in V_0[\Lambda_0 - (\varepsilon_{r_1} - \delta_1) - \cdots - (\varepsilon_{r_k} - \delta_1)] \qquad v \in V_0(\Lambda_0). \tag{35}$$

In discussing shift components of antisymmetric kth-rank contragredient tensors, it is convenient to introduce the antisymmetric tensor matrix

$$a^{i_1 i_2 \ldots i_k}_{j_1 j_2 \ldots j_k} \tag{36}$$

whose matrix powers are defined for $m \geq 0$ by (Green, 1971):

$$(a^{m+1})^{i_1 i_2 \ldots i_k}_{j_1 j_2 \ldots j_k} = a^{i_1}_q (a^m)^{q i_2 \ldots i_k}_{j_1 j_2 \ldots j_k} + a^{i_2}_q (a^m)^{i_1 q \ldots i_k}_{j_1 j_2 \ldots j_k} + \ldots + a^{i_k}_q (a^m)^{i_1 \ldots i_{k-1} q}_{j_1 \ldots j_{k-1} j_k}$$

with

$$\begin{aligned}(a^0)^{i_1 i_2 \ldots i_k}_{j_1 j_2 \ldots j_k} &= \delta^{i_1 i_2 \ldots i_k}_{j_1 j_2 \ldots j_k} \\ &= \frac{1}{k!} \sum_\pi \mathrm{sn}(\pi) \delta^{i_{\pi(1)}}_{j_1} \delta^{i_{\pi(2)}}_{j_2} \ldots \delta^{i_{\pi(k)}}_{j_k}.\end{aligned}$$

Thus, for example, for the case $k = 2$ we have

$$\begin{aligned}\delta^{ij}_{kl} &= \tfrac{1}{2}(\delta^i_k \delta^j_l - \delta^i_l \delta^j_k) \\ a^{ij}_{kl} &= \tfrac{1}{2}(\delta^i_k a^j_l + \delta^j_l a^i_k - \delta^j_k a^i_l - \delta^i_l a^j_k)\end{aligned}$$

etc. Then we may construct the appropriate tensor projections in terms of the tensor matrix a, as follows $(1 \leq r_1 < r_2 < \ldots < r_k \leq n)$:

$$P[r_1, r_2, \ldots, r_k] = \prod_{1 \leq l_1 < l_2 < \ldots < l_k \leq n} \left(\frac{a - \alpha_{l_1, l_2, \ldots, l_k}}{\alpha_{r_1, \ldots, r_k} - \alpha_{l_1, \ldots, l_k}} \right) \tag{37}$$

where the product is over all ascending sequences of k numbers $1 \leq l_1 < l_2 \ldots < l_k \leq n$, except for $(l_1, l_2, \ldots, l_k) = (r_1, r_2, \ldots, r_k)$, and the characteristic tensor roots are given by (Green 1971)

$$\alpha_{l_1, l_2, \ldots, l_k} = \alpha_{l_1} + \alpha_{l_2} + \ldots + \alpha_{l_k} + \tfrac{1}{2}k(k + 1 - 2n)$$

with α_r as in (32).

In terms of the tensor matrices (37), if $\Psi_{i_1 \ldots i_k}$ is an arbitrary antisymmetric kth rank contragredient tensor, then its shift components are given by

$$\begin{aligned}\Psi_{i_1 \ldots i_k} &= \sum_{1 \leq r_1 < r_2 < \ldots < r_k \leq n} \Psi[r_1, r_2, \ldots, r_k]_{i_1 \ldots i_k} \\ \Psi[r_1, r_2, \ldots, r_k]_{i_1 \ldots i_k} &= \Psi_{j_1 j_2 \ldots j_k} P[r_1, r_2, \ldots, r_k]^{j_1 j_2 \ldots j_k}_{i_1 i_2 \ldots i_k}.\end{aligned} \tag{38}$$

In this notation (34) may be expressed as

$$\begin{aligned}\Psi[r_1, r_2, \ldots, r_k]_{i_1 i_2 \ldots i_k} &= \Psi[r_1 : r_2, \ldots, r_k]_{i_1 \ldots i_k} + \Psi[r_2 : r_1, r_3, \ldots, r_k]_{i_1 \ldots i_k} + \ldots \\ &\quad + \Psi[r_k : r_1, r_2, \ldots, r_{k-1}]_{i_1 \ldots i_k}\end{aligned} \tag{39}$$

where the 'primary' shift components on the RHS are given by

$$\begin{aligned}\Psi[r_1:r_2,\ldots,r_k]_{i_1\ldots i_k} &= \Psi[r_1]_{i_1}\Psi[r_2,\ldots,r_k]_{i_2\ldots i_k}\\ &= \bar{P}[r_1]_{i_1}{}^{j}\Psi_j\Psi_{j_2}\ldots\Psi_{j_k}P[r_2,\ldots,r_k]^{j_2\ldots j_k}_{i_2\ldots i_k}.\end{aligned}$$

Equation (39) in fact extends to arbitrary antisymmetric contragredient kth-rank tensors, whose primary shift components are given, in general, by

$$\Psi[r_1:r_2,\ldots,r_k]_{i_1\ldots i_k} = \bar{P}[r_1]^{j}_{i_1}\Psi_{jj_2\ldots j_k}P[r_2,\ldots,r_k]^{j_2\ldots j_k}_{i_2\ldots i_k}. \tag{40}$$

The shift components (38) and primary shift components (40) effect shifts on the $\mathrm{gl}(n)\oplus\mathrm{gl}(1)$ representation labels, given by (32).

The above shift component analysis is ideally suited to investigate the nature of the irreducible $\mathrm{gl}(n|1)$ modules $V(\Lambda)$, using the modified induced module construction of (16), to which we now turn.

We have, using (22) and (31), for $v\in V_0(\Lambda-2\rho_1)$,

$$\begin{aligned}\Psi[r]_i\bar{\Psi}\otimes v &= \Psi_j\bar{\Psi}\otimes P[r]^j_i v\\ &= \bar{\Psi}_k\otimes(a+\Omega-n+1)^k_j P[r]^j_i v\\ &= \bar{\Psi}[r]_k(\alpha_r+\Omega-n+1)v.\end{aligned}$$

We note that the factor $\alpha_r+\Omega-n+1$ is given by

$$\Lambda'_r+n-r+\omega'-n+1$$

where Λ'_i, ω' refer to the components of $\Lambda'=\Lambda-2\rho_1$. We thus have, in terms of the highest-weight labels $\Lambda=(\Lambda_1,\ldots,\Lambda_n|\omega)$,

$$\Lambda'_r=\Lambda_r-1 \qquad \omega'=\omega+n$$

from which we obtain

$$\begin{aligned}\Psi[r]_i\bar{\Psi}\otimes v &= (\Lambda+\rho,\,\varepsilon_r-\delta_1)\bar{\Psi}[r]_i v\\ &= (\Lambda+\rho,\,\varepsilon_r-\delta_1)\bar{\Psi}_j\otimes P[r]^j_i v\end{aligned}$$

where we have used the result:

$$(\Lambda+\rho,\,\varepsilon_r-\delta_1)=\Lambda_r+\omega+n-r.$$

The above shows immediately that the irreducible $\mathrm{gl}(n)\oplus\mathrm{gl}(1)$ module

$$V_0(\Lambda-\varepsilon_r+\delta_1)$$

only occurs in $V(\Lambda)$ provided $(\Lambda+\rho,\,\varepsilon_r-\delta_1)\neq 0$.

Proceeding inductively suppose, for $1\leqslant r_1<r_2<\ldots<r_k\leqslant n$, that the action of the shift components of the antisymmetric contragredient tensor $\Psi_{i_1}\Psi_{i_2}\ldots\Psi_{i_k}$ is given by

$$\Psi[r_1,r_2,\ldots r_k]_{i_1\ldots i_k}\bar{\Psi}\otimes v=\prod_{i=1}^{k}(\Lambda+\rho,\,\varepsilon_{r_i}-\delta_1)\bar{\Psi}_{j_1\ldots j_k}\otimes P[r_1,\ldots,r_k]^{j_1\ldots j_k}_{i_1\ldots i_k}v. \tag{41}$$

Multiplying on the left by $\Psi[r]_i$ we obtain

$$\Psi[r]_i\Psi[r_1,\ldots,r_k]_{i_1\ldots i_k}\bar{\Psi}\otimes v=\prod_{i=1}^{k}(\Lambda+\rho,\,\varepsilon_{r_i}-\delta_1)\bar{P}[r]^j_i\Psi_j\bar{\Psi}_{j_1\ldots j_k}\otimes P[r_1,\ldots,r_k]^{j_1\ldots j_k}_{i_1\ldots i_k}v.$$

Using (28) we obtain, in terms of the primary shift components of (38), for the RHS above:

$$\prod_{i=1}^{k} (\Lambda+\rho, \varepsilon_{r_i}-\delta_1)(\Omega+n-1-k-\bar{\alpha}_r)\bar{\Psi}[r:r_1,\ldots,r_k]_{ii_1\ldots i_k} v$$

where we have employed (33). In this case (cf (32))

$$\Omega+n-1-k-\bar{\alpha}_r=\omega'+n-1-k-(r-1-\Lambda'_r)$$

where Λ'_r, ω' now refer to the $\mathrm{gl}(n)\oplus\mathrm{gl}(1)$ components of the shifted weight:

$$\Lambda'=\Lambda-(\varepsilon_r-\delta_1)-\sum_{i=1}^{k}(\varepsilon_{r_i}-\delta_1).$$

Thus we obtain, in terms of highest-weight labels,

$$\Omega+n-1-k-\bar{\alpha}_r=\Lambda_r+\omega+n-r=(\Lambda+\rho, \varepsilon_r-\delta_1)$$

giving

$$\Psi[r]_i\Psi[r_1,\ldots,r_k]_{i_1\ldots i_k}\bar{\Psi}\otimes v$$
$$=(\Lambda+\rho, \varepsilon_r-\delta_1)\prod_{i=1}^{k}(\Lambda+\rho, \varepsilon_{r_i}-\delta_1)\bar{\Psi}[r:r_1,\ldots,r_k]_{ii_1\ldots i_k} v.$$

Finally, by symmetrising the shift labels using (39), we obtain the following pure shift antisymmetric tensor equation:

$$\Psi[r,r_1,\ldots,r_k]_{ii_1\ldots i_k}\bar{\Psi}\otimes v$$
$$=(\Lambda+\rho, \varepsilon_r-\delta_1)\prod_{i=1}^{k}(\Lambda+\rho, \varepsilon_{r_i}-\delta_1)\bar{\Psi}[r,r_1,\ldots,r_k]_{ii_1\ldots i_k} v.$$

Thus, by induction, we have established (41) for all values of k, $1\leqslant k\leqslant n$.

The vectors on the LHS of (41) either span the irreducible $\mathrm{gl}(n)\oplus\mathrm{gl}(1)$ submodule of $V(\Lambda)$ with highest weight

$$\Lambda-\sum_{i=1}^{k}(\varepsilon_{r_i}-\delta_1) \tag{42}$$

or else are all zero for every $v\in V_0(\Lambda-2\rho_1)$ and choice of indices $1\leqslant i_1, i_2,\ldots,i_k\leqslant n$. In view of (41), the latter case only occurs if the weight (42) is not dominant or if $(\Lambda+\rho, \varepsilon_{r_i}-\delta_1)=0$ for some i ($\leqslant k$). Moreover, in view of the fact that $\Lambda\in D^+$, it is easily seen that $(\Lambda+\rho, \alpha)$ can be zero for at most one odd positive root $\alpha=\varepsilon_i-\delta_1$ (i.e. the atypical irreducible representations of $\mathrm{gl}(n|1)$ are all singly atypical, as noted before): in such a case we refer to the index i as the Λ-atypical index (the remaining indices are referred to as Λ-typical). This suggests that we introduce the set of Λ-typical odd positive roots

$$\Phi_1^+(\Lambda)=\{\alpha\in\Phi_1^+|(\Lambda+\rho, \alpha)\neq 0\}.$$

In view of the above remarks we have $\Phi_1^+(\Lambda)=\Phi_1^+$ for $\Lambda\in D^+$ typical, whilst for Λ atypical $|\Phi_1^+(\Lambda)|=|\Phi_1^+|-1$. We now let $P(\Lambda)$ denote the power set of $\Phi_1^+(\Lambda)$ (i.e. $P(\Lambda)$ is the set of subsets of $\Phi_1^+(\Lambda)$) and for any subset $\theta\subseteq\Phi_1^+(\Lambda)$ we write

$$\rho_1(\theta)=\tfrac{1}{2}\sum_{\alpha\in\theta}\alpha \tag{43}$$

with $\rho_1(\varnothing)=0$ ($\varnothing$ the empty set). With this convention we have the following result.

Theorem 2. The finite-dimensional irreducible $\mathrm{gl}(n|1)$ module $V(\Lambda)$ is the direct sum of all finite-dimensional irreducible $\mathrm{gl}(n)\oplus\mathrm{gl}(1)$ modules with highest weights

$$\Lambda_0 = \Lambda - 2\rho_1(\theta) \qquad \theta \in P(\Lambda)$$

subject to $\Lambda_0 \in D^+$, each occurring with multiplicity 1.

Another way of phrasing the above result is to note that the $\mathrm{gl}(n)\oplus\mathrm{gl}(1)$ highest weights $\Lambda_0 = (\Lambda_{0_1}, \ldots, \Lambda_{0_n}|\omega_0)$ occurring in $V(\Lambda)$ are to satisfy

$$\begin{aligned} &\omega_0 = \omega + \sum_{i=1}^{n} (\Lambda_i - \Lambda_{0_i}) \\ &\left.\begin{aligned} &\Lambda_i \geqslant \Lambda_{0_i} \geqslant \Lambda_i - 1 && 1 \leqslant i \leqslant n \\ &\Lambda_{0_i} \geqslant \Lambda_{0_{i+1}} && 1 \leqslant i \leqslant n-1 \end{aligned}\right\} \quad \text{if } \Lambda_i + \omega \neq i - n \\ &\Lambda_{0_i} = \Lambda_i \qquad\qquad\qquad\qquad\qquad\quad \text{if } \Lambda_i + \omega = i - n. \end{aligned} \tag{44}$$

In addition, every $\Lambda_i - \Lambda_{0_i}$ must be integral. Each allowed $V_0(\Lambda_0)$ occurs in $V(\Lambda)$ with unit multiplicity. In the case that Λ is typical, this branching rule obviously reduces to that given by (14) as required. These rules are consistent with those implicit in the results of Palev (1987, 1988a, b).

We conclude this section with some comments on the weight spectrum of $V(\Lambda)$. We note that the $\mathrm{gl}(n)\oplus\mathrm{gl}(1)$ highest weights occurring in $V(\Lambda)$ are obtained from Λ by decreasing the components Λ_i by at most one unit when i is Λ-typical, i.e. $\Lambda_i + \omega \neq i - n$. However, even if the index i is Λ-typical we cannot decrease Λ_i if there exists a Λ-atypical index $j > i$ with $\Lambda_j = \Lambda_i$ (i.e. $(\Lambda, \varepsilon_i - \varepsilon_j) = 0$). (This eventuality is automatically taken into account in theorem 2 by requiring the highest weights Λ_0 to be dominant integral). This leads us to consider the modified set of odd positive roots:

$$\tilde{\Phi}_1^+(\Lambda) = \{\alpha \in \Phi_1^+(\Lambda) | \beta \in \Phi_1^+ \qquad \text{with} \quad \alpha - \beta \in \Phi_0^+$$

and

$$(\Lambda, \alpha - \beta) = 0 \Rightarrow \beta \in \Phi_1^+(\Lambda)\}$$

whose power set, denoted $\tilde{P}(\Lambda)$, may replace $P(\Lambda)$ in theorem 2: we have the corresponding index set

$$I_\Lambda = \{1 \leqslant i \leqslant n | (\varepsilon_i - \delta_1) \in \tilde{\Phi}_1^+(\Lambda)\}.$$

Finally we set (notation as in (43))

$$\rho_1(\Lambda) = \rho_1(\tilde{\Phi}_1^+(\Lambda)).$$

Then it is easily seen that $\Lambda - 2\rho_1(\Lambda) \in D^+$ and hence, by theorem 2, the irreducible $\mathrm{gl}(n)\oplus\mathrm{gl}(1)$ module with highest weight

$$\Lambda - 2\rho_1(\Lambda)$$

necessarily occurs in $V(\Lambda)$, and is characterised by the fact that it is minimal among the set of $\mathrm{gl}(n)\oplus\mathrm{gl}(1)$ highest weights in $V(\Lambda)$.

It follows that the minimal weight of $V(\Lambda)$ is given by

$$\Lambda^{(-)} = \tau(\Lambda - 2\rho_1(\Lambda))$$

where τ is the unique Weyl group element sending positive roots to negative roots (Humphreys 1972). We thus arrive at the following result.

Lemma (notation as above). Let $V(\Lambda)$ be a finite-dimensional irreducible gl$(n|1)$ module with highest weight $\Lambda \in D^+$. Then

(i) the minimal weight of $V(\Lambda)$ is

$$\Lambda^{(-)} = \Lambda_- - \sum_{i \in I_\Lambda} (\varepsilon_{n+1-i} - \delta_1)$$

where Λ_- is the minimal weight of the gl$(n)\oplus$gl(1) module $V_0(\Lambda)$, and

(ii) for every weight ν in $V(\Lambda)$ s.t. $\nu \neq \Lambda, \Lambda^{(-)}$, we have

$$\Lambda > \nu > \Lambda^{(-)}.$$

Remark. The ordering $>$ referred to above is the natural ordering induced on the weights by the (distinguished) system of simple roots: we write $\lambda > \mu$ if and only if $\lambda \neq \mu$ and

$$\lambda - \mu = \sum_{i=1}^{n-1} n_i(\varepsilon_i - \varepsilon_{i+1}) + n_s\alpha_s$$

with n_i, $n_s \in \mathbb{Z}^+$. We note also that the weight spectrum of $V(\Lambda)$ is stable under the action of the Weyl group of gl$(n)\oplus$gl(1).

5. Character formula for gl$(n|1)$

As an application of our results, we present a confirmation for gl$(n|1)$ of the character formula proposed by Hughes and King (1987).

To write the branching rule of the previous section in a convenient form for our present purposes, we introduce the index set

$$\tilde{I}_\Lambda = \{i \in I_\Lambda | (\Lambda, \varepsilon_i - \varepsilon_{i+1}) \neq 0\}.$$

For each $i \in \tilde{I}_\Lambda$ we let m_i be the largest non-negative integer such that

$$i - m_i \in I_\Lambda \qquad (\Lambda, \varepsilon_i - \varepsilon_{i-m_i}) = 0.$$

With this notation the index set I_Λ of § 4 is given by

$$I_\Lambda = \bigcup_{i \in \tilde{I}_\Lambda} \{i, i-1, \ldots, i-m_i\}$$

and the gl$(n|1)\downarrow$gl$(n)\oplus$gl(1) branching rule may be written

$$V(\Lambda) = \oplus' V_0(\Lambda_0) \tag{45}$$

where the gl$(n)\oplus$gl(1) representation labels Λ_0 are of the form

$$\Lambda_0 = \Lambda - \sum_{i \in \tilde{I}_\Lambda} n_i[\varepsilon_i + \varepsilon_{i-1} + \cdots + \varepsilon_{i-k_i} - (k_i+1)\delta_1] \tag{46}$$

and the sum in (46) is over all sets of integers $n_i = 0$, 1 and $0 \leq k_i \leq m_i$ $(i \in \tilde{I}_\Lambda)$. It follows that the number of irreducible gl$(n)\oplus$gl(1) modules occurring in the decomposition (45) is given by the integer

$$2^{|\tilde{I}_\Lambda|} \prod_{i \in \tilde{I}_\Lambda} (m_i + 1).$$

Now let W be the Weyl group of gl(n). Applying the usual gl(n) character formula (Humphreys 1972) we obtain the following formula for the character of $V(\Lambda)$:

$$q \text{ ch } V(\Lambda) = \sum_{\sigma \in W} \text{sn}\, \sigma \sum_{\Lambda_0}{}' \exp[\sigma(\Lambda_0 + \rho_0)]$$

where the sum over Λ_0 is over all $\mathrm{gl}(n)\oplus\mathrm{gl}(1)$ highest weights occurring in $V(\Lambda)$ and q denotes the usual Weyl denominator function:

$$q=\prod_{\alpha\in\Phi_0^+}(e^{\alpha/2}-e^{-\alpha/2}).$$

Using the fact that $\sigma(\rho_1)=\rho_1$ for all $\sigma\in W$, together with (46), we see that the above character formula reduces to

$$q\ \mathrm{ch}\ V(\Lambda)=e^{\rho_1}\sum_{\sigma\in W}\mathrm{sn}\ \sigma\ e^{\sigma(\Lambda+\rho)}\prod_{i\in\bar{I}_\Lambda}\theta_i^\sigma \tag{47}$$

where

$$\theta_i^\sigma=1+\exp[-\sigma(\varepsilon_i-\delta_1)]+\exp[-\sigma(\varepsilon_i+\varepsilon_{i-1}-2\delta_1)]+\ldots+\exp\left[-\sigma\left(\sum_{j=i}^{i-m_i}(\varepsilon_j-\delta_1)\right)\right]. \tag{48}$$

To simplify (47) we write

$$q\ \mathrm{ch}\ V(\Lambda)=e^{\rho_1}\sum_{\sigma\in W}\mathrm{sn}\ \sigma\ e^{\sigma(\Lambda+\rho)}\ \theta_i^\sigma\tilde{\theta}_i^\sigma$$

where

$$\tilde{\theta}_i^\sigma=\prod_{\substack{k\in\bar{I}_\Lambda\\ k\neq i}}\theta_k^\sigma.$$

Now for each index j s.t. $i-m_i\leqslant j<i$ consider the reflection $\sigma_j\in W$ determined by the root $\varepsilon_j-\varepsilon_{j+1}$:

$$\sigma_j(\Lambda)=\Lambda-(\Lambda,\varepsilon_j-\varepsilon_{j+1})(\varepsilon_j-\varepsilon_{j+1}).$$

Then we have

$$\begin{aligned}
&\sum_{\sigma\in W}\mathrm{sn}\ \sigma\exp[\sigma(\Lambda+\rho)]\exp[-\sigma(\varepsilon_j-\delta_1)]\tilde{\theta}_i^\sigma\\
&\quad=\sum_{\sigma\in W}\mathrm{sn}(\sigma\sigma_j)\exp[\sigma\sigma_j(\Lambda+\rho)]\exp[-\sigma\sigma_j(\varepsilon_j-\delta_1)]\tilde{\theta}_i^{\sigma\sigma_j}\\
&\quad=-\sum_{\sigma\in W}\mathrm{sn}\ \sigma\exp[\sigma(\Lambda+\rho)]\exp[-\sigma(\varepsilon_j-\varepsilon_{j+1})]\exp[-\sigma(\varepsilon_{j+1}-\delta_1)]\tilde{\theta}_i^\sigma\\
&\quad=-\sum_{\sigma\in W}\mathrm{sn}\ \sigma\exp[\sigma(\Lambda+\rho)]\exp[-\sigma(\varepsilon_j-\delta_1)]\tilde{\theta}_i^\sigma=0
\end{aligned}$$

where, in the above, we used the fact that $\tilde{\theta}_i^{\sigma\sigma_j}=\tilde{\theta}_i^\sigma$ together with

$$\sigma_j(\Lambda+\rho)=\Lambda+\rho-(\varepsilon_j-\varepsilon_{j+1})$$

which follows from the relations

$$(\Lambda,\varepsilon_j-\varepsilon_{j+1})=0\qquad(\rho,\varepsilon_j-\varepsilon_{j+1})=(\rho_0,\varepsilon_j-\varepsilon_{j+1})=1.$$

It follows that the term $\exp[-\sigma(\varepsilon_i-\delta_1)]$ occurring in (48) may be replaced by the symmetrised sum

$$\sum_{j=i}^{i-m_i}\exp[-\sigma(\varepsilon_j-\delta_1)]$$

since the remaining terms contribute zero to the Weyl group sum. Similarly, if j, k are two indices such that $i-m_i \leq j < k < i$, or $k=i$ and $j<i-1$, then by applying the Weyl group reflections σ_j, σ_k we deduce

$$\sum_{\sigma} \operatorname{sn}\sigma \exp[\sigma(\Lambda+\rho)] \exp[-\sigma(\varepsilon_j+\varepsilon_k-2\delta_1)]\tilde{\theta}_i^{\sigma}=0.$$

We may thus replace the term $\exp[-\sigma(\varepsilon_i+\varepsilon_{i-1}-2\delta_1)]$ occurring in (48) with the sum

$$\sum_{i-m_i\leq j<k\leq i} \exp[-\sigma(\varepsilon_j+\varepsilon_k-2\delta_1)].$$

Proceeding in this way we deduce that, in (47), θ_i^{σ} may be replaced with the symmetrised expression

$$\begin{aligned}\theta_i^{\sigma} &= 1+\sum_{i-m_i\leq j\leq i} \exp[-\sigma(\varepsilon_j-\delta_1)]+\sum_{i-m_i\leq j<k\leq i} \exp[-\sigma(\varepsilon_j+\varepsilon_k-2\delta_1)]+\ldots \\ &\quad +\exp\left[-\sigma\left(\sum_{i-m_i\leq j\leq i}(\varepsilon_j-\delta_1)\right)\right] \\ &= \prod_{j=i}^{i-m_i}{}' \{1+\exp[-\sigma(\varepsilon_j-\delta_1)]\}.\end{aligned}$$

We thus arrive at the character formula:

$$q \operatorname{ch} V(\Lambda)=\mathrm{e}^{\rho_1} \sum_{\sigma\in W} \operatorname{sn}\sigma \exp[\sigma(\Lambda+\rho)] \prod_{i\in \bar{I}_\Lambda} \prod_{j=i}^{i-m_i}{}' \{1+\exp[-\sigma(\varepsilon_j-\delta_1)]\}$$

which, in terms of the index set I_Λ of § 4, simplifies to

$$\begin{aligned}q \operatorname{ch} V(\Lambda) &= \mathrm{e}^{\rho_1} \sum_{\sigma\in W} \operatorname{sn}\sigma \exp[\sigma(\Lambda+\rho)] \prod_{i\in I_\Lambda} \{1+\exp[-\sigma(\varepsilon_i-\delta_1)]\} \\ &= \mathrm{e}^{\rho_1} \sum_{\sigma\in W} \operatorname{sn}\sigma \exp[\sigma(\Lambda+\rho)] \prod_{\alpha\in\tilde{\Phi}_1^+(\Lambda)} \{1+\exp[-\sigma(\alpha)]\}.\end{aligned}$$

This is the character formula conjectured by Hughes and King (1987). It differs from the formula proposed by Bernstein and Leites (1980) for $\mathrm{gl}(n|m)$ (and later restricted to $\mathrm{gl}(n|1)$ (Leites 1987)) and by Van der Jeugt (1987) for singly atypical representations of $\mathrm{sl}(n|m)$, which here would have $\tilde{\Phi}_1^+(\Lambda)$ replaced by $\Phi_1^+(\Lambda)$. It is possible that the two formulae are equivalent for $\mathrm{gl}(n|1)$, but this remains to be proved.

6. Discussion

The structure and characters of all irreducible finite-dimensional $\mathrm{gl}(n|1)$ modules have been determined. From the branching rule of theorem 2, we could explicitly construct an (orthonormal) super Gel'fand–Tsetlin basis for the irreducible modules, which is symmetry adapted to the subalgebra chain

$$\mathrm{gl}(n|1)\supset \mathrm{gl}(n)\supset \mathrm{gl}(n-1)\supset\cdots \mathrm{gl}(1).$$

It is of great interest to know the matrix elements of the $\mathrm{gl}(n|1)$ generators in such a basis, and to characterise the star and grade-star representations (Scheunert *et al* 1977) of this superalgebra. The former problem has already been investigated, for both typical and atypical cases, by Palev (1987, 1988a, b).

It would also be of interest to extend the approach of this paper to gl($n|m$) (at least for determining the gl(n)⊕gl(m) content and hence characters of the irreducible representations). There is evidence to suggest that our approach will extend to this case, in the framework of antisymmetric tensors of gl(n)⊕gl(m), since the induced module construction of (16) still applies (Gould 1988a) as it does for the Lie superalgebras C(n). In the case of type-II simple Lie superalgebras the situation is more complex because the construction of (16) no longer applies, so that a suitable generalisation would be required.

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